AD-A186 365

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION	1b. RESTRICTIVE MARKINGS
UNCLASSIFIED	

2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION AVAILABILITY OF REPORT
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	Approved for public release; distribution unlimited.

4. PERFORMING ORGANIZATION REPORT NUMBER(S)	5. MONITORING ORGANIZATION REPORT NUMBER(S)
	AFOSR-TR- 87 - 1 2 5 0

6a. NAME OF PERFORMING ORGANIZATION	6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION
Columbia University		Air Force Office of Scientific Research

6c. ADDRESS (City, State and ZIP Code)	7b. ADDRESS (City, State and ZIP Code)
Columbia University New York, NY 10027	Directorate of Mathematical & Information Sciences, Bolling AFB DC 20332-6448

8a. NAME OF FUNDING/SPONSORING ORGANIZATION	8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
AFOSR	NM	AFOSR 87-0072

8c. ADDRESS (City, State and ZIP Code)	10. SOURCE OF FUNDING NOS.			
Bld 410 Bolling AFB DC 20332-6448	PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
	61102F	2304	A5	

11. TITLE (Include Security Classification)
On Stochastic Optimality of Policies in first Passage Problems

12. PERSONAL AUTHOR(S)
Michael N. Katehakis and Costis Melolidakis

13a. TYPE OF REPORT	13b. TIME COVERED FROM _____ TO _____	14. DATE OF REPORT (Yr, Mo, Day)	15. PAGE COUNT

16. SUPPLEMENTARY NOTATION

17.	COSATI CODES		18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB GP	

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

In stochastic scheduling and optimal maintenance problems that have been considered in the literature, the optimization criterion used has often been equivalent to minimizing the expected first passage times to a set of states. A typical method used in establishing the optimality of a certain policy is the method of successive approximations on the appropriate dynamic programming functional equations. As an intermediate result, this technique often involves the optimality of the pertinent policy for all finite horizon versions of the problem. In this paper we characterize stochastically optimal policies as policies that process a similar property, i.e. they are optimal in expectation for all members of a sequence of appropriately defined finite horizon problems. We use this characterization to establish the stochastic optimality of relevant policies for the optimal repair allocation for a series system problem and for a scheluding problem.
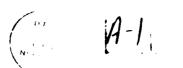
20. DISTRIBUTION/AVAILABILITY OF ABSTRACT	21. ABSTRACT SECURITY CLASSIFICATION
UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT ☐ DTIC USERS ☐	UNCLASSIFIED

22a. NAME OF RESPONSIBLE INDIVIDUAL	22b. TELEPHONE NUMBER (Include Area Code)	22c. OFFICE SYMBOL
B. Woodruff	(202) 767- 5027	NM

DD FORM 1473, 83 APR   EDITION OF 1 JAN 73 IS OBSOLETE   UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

ON STOCHASTIC OPTIMALITY OF POLICIES IN FIRST PASSAGE PROBLEMS[*]

by

Michael N. Katehakis   and   Costis Melolidakis
Technical Univesrity of Crete   Technical University of Crete
and Columbia University

ABSTRACT

In stochastic scheduling and optimal maintenance problems that have
been considered in the literature, the optimization criterion used has
often been equivalent to minimizing the expected first passage times to a
set of states. A typical method used in establishing the optimality of a
certain policy is the method of successive approximations on the
appropriate dynamic programming functional equations. As an
intermediate result, this technique often involves the optimality of the
pertinent policy for all finite horizon versions of the problem. In this
paper we characterize stochastically optimal policies as policies that
posess a similar property, i.e. they are optimal in expectation for all
members of a sequence of appropriately defined finite horizon problems.
We use this characterization to establish the stochastic optimality of
relevant policies for the optimal repair allocation for a series system
problem and for a scheluling problem.

1. Introduction. In many problems that have been considered in the litera-

ture of stochastic scheduling and maintenance, the optimization criterion

employed has often been to minimize the expected first passage times to a set

of "desirable" states. A typical method used in establishing the optimality of a

certain policy is the method of successive approximations on the appropriate

dynamic programming functional equations. As an intermediate result, this

technique often involves the optimality of the pertinent policy for all finite

horizon versions of the problem.

In this paper we show that the stochastic optimality of a policy can be obtained in a similar manner by establishing that the policy is optimal for a class of appropriately defined finite horizon problems. Furthermore this approach can be used to establish that a stochastically optimal policy does not exist. It is known that stochastic optimality is the strongest optimization criterion since it implies optimality under both the expected and the discounted first passage time criteria.

In this paper we examine two cases of application of the property above.

We first consider a generalization of the problem of optimal allocation over time of a single repairman to failed components of a series system, previously considered in Katehakis and Derman (1984); see this paper for references on other work on this problem. Operation in a varying enviroment is considered and the following assumptions are made. Let N denote the number of components in the system . Let θ denote the state of the environment which is observable and let Θ denote the set of all possible states of the environment; Θ is assumed to be (for simplicity) finite. Furthermore, we model the law of motion for the state of the environment by a continuous time Markov Chain $\{\theta(t) , t \geqslant 0\}$ with known transition rates $\{q(\theta'/\theta), \theta,\theta' \in \Theta\}$. Components may be either in a functioning or in a failed state. To model the effect of the operating environment on the time to failure of the components, we assume that when the state of the environment is θ the failure time of the ith component is an exponentially distributed random variable with known rate $\mu_i(\theta)$ ($1 \leqslant i \leqslant N$, $\theta \in \Theta$) . In addition we assume that the failure time of any component is independent of the state of other components. The time required to repair component i is also an exponentially distributed random variable the rate λ_i of which is independent of the state of the environment. Repaired components are as good as new. It is assumed that it

2

A-1

is possible to reassign the repairman among failed components instantaneously.

In this paper it is shown that the policy which always assigns the repairman to the failed component with the smallest failure rate among the failed ones (smallest failure rate first or SFR policy) minimizes stochastically the time the system spends under repair. Hence, it is optimal with respect to both the maximum availability and the maximum discounted operation time criteria, irrespective of the values of the repair rates and the discount rate.

In Katehakis and Derman (1984), the optimality of the SFR policy with respect to the average system operation time criterion was obtained in the case of non-varying enviroment. The proof involved establishing that the SFR policy minimized the expected first passage times to the functioning state; this was done by showing that the functional equations of the relevant Markovian decision problem were valid under this policy. The method of proof then was based on induction on the number of the components. The present proof, based on propositions 1 and 2 below, although simpler than the original, establishes optimality under a stronger criterion for a more general model.

We also consider the following stochastic scheduling problem examined by Van der Heyden (1981); see also Weiss and Pinedo (1982). Jobs arrive according to a Poisson process with rate r. The processing time of a job is an exponentially distributed random variable with a parameter that is chosen upon arrival by sampling from a known distribution F. All random variables are assumed to be independent. There are p processors and the objective is to minimize the expected time until all jobs have been completed (also called the expected makespan). It has been established, in the papers above, that the policy which always assigns processors to the uncompleted jobs with the Longest Expected Processing Times (LEPT policy) minimizes the expected

3

makespan. We modify Van der Heyden's work and obtain the stochastic

optimality of the LEPT policy.

2. Characterization of Stochastically Optimal Policies in First Passage Problems.

We first consider the discrete time first passage problem in Markovian

Decision Theory on a (for simplicity) countable state space, which is specified

by the following elements.

1. The state space S and the action sets $A(s)$, $s \in S$,

2. The transition law: $\{p(s'/s,a)$, $s',s \in S$, $a \in A(s)\}$,

3. A subset S_0 of S ,

4. An initial state s_0 .

We will denote this problem by (Π_d) . A policy π generates a stochastic

process $\{X_\pi(k)$, $k = 1,2,...\}$. The first passage time from a state s to S_0

will be denoted by $T_\pi(s)$.

A policy π^0 is called stochastically optimal if it satisfies

(1) $T_{\pi 0}(s) \overset{st}{\text{≤}} T_\pi(s)$ for all alternative policies π , $s \in S$

where, given two random variables Y_1 , Y_2 , define:

(2) $Y_1 \overset{st}{\text{≤}} Y_2$ if and only if $P(Y_1 \leq y) \geq P(Y_2 \leq y)$.

The method we use to show the stochastic optimality of a policy and to

discover whether such a policy exists is based on establishing the optimality

of the policy in the following class of finite horizon problems.

We define the finite horizon problem (Π_n) , $n \geq 1$, as follows.

1. State space $S_n = \{(s:m)$, $s \in S$, $m = 0,1,...,n\}$.

2. Action sets: $A(s:m) = A(s))$ $m = 1,...,n$, $A(s:0) = 0$

3. Transition law:

$$
(3) \quad p((s';m')/(s;m),a) = \begin{cases} p(s'/s,a) & \text{if } m' = m - 1 \text{ and } s , s' \in S \setminus S_0 \\ 1 & \text{if } m' = m - 1 \text{ and } s , s' \in S_0 \\ 0 & \text{otherwise .} \end{cases}
$$

4. Reward structure:

$$
(4) \quad r(s;m) = \begin{cases} 1 & \text{if } s \in S_0 \text{ and } m = 0 \\ 0 & \text{otherwise .} \end{cases}
$$

Every policy in the process (Π) induces a policy refering to the family of processes $\{(\Pi_n) , n \geq 1\}$ which does not depend on n and vise versa. Thus, there is a $1 - 1$ correspondence between policies associated with the problem (Π) and policies refering to the family of problems $\{(\Pi_n)\}$ which do not depend on n .

The following can be easily established.

Proposition 1. A policy is stochastically optimal in (Π) if and only if it is optimal in (Π_n) for all $n \geq 1$.

Proof: It suffices to notice that in n steps the process either terminates in the set of states $\{(s;0) : s \in S_0\}$ with reward 1 or to some other state with reward 0 . Thus, the total expected reward in (Π_n) coresponding to any policy π defined in (Π) and initial state (s,n) is $P[T_\pi(s) \leq n]$.

For a continuous time first passage problem the approach described above is applicable if we can use the device of uniformization; see Jensen(1953), Veinott(1969) and Lipman(1975). To be precise, the continuous time problem, which is denoted by (Π_c) , is specified by:

1. The state space S and the action sets $A(s)$, $s \in S$,

5

2. The transition law, which is given in terms of transition rates

   $\{v(s'/s,a)$, s' , $s \in S$, $a \in A(s)\}$,

3. A subset S_0 of S ,

4. An initial state s_0 .

   Let us define

$$(5) \quad v(s,a) = \sum_{s'} v(s'/s,a)$$

The device of uniformization essentially involves to notice that if the transition rates are bounded and if we consider $v \geq v(s,a)$ $\forall$ s, a , then, by counting (dummy) transitions back to state s at a rate $(v - v(s,a))$ the sojourn times in all states are equalized, i.e., they are i.i.d. exponentially distributed random variables with rate v . Thus, a discrete first passage problem is defined on the same state and action spaces with transition probabilities given by:

$$(6) \quad p(s'/s,a) = \begin{cases} v(s'/s,a)/v & \text{if } s \neq s' \\ (v - v(s'/s,a))/v & \text{if } s = s' \end{cases}$$

Let $T_\pi^C(s)$ denote the first passage time from state s to the subset S_0 for the original continuous time process and let $T_\pi^-(s)$ denote the first passage time from s to S_0 for the discrete time process above (note that without any loss in generality we can assume that $v = 1$ and thus regard $T_\pi^-(s)$ as a random variable counting the number of transitions to S_0) .

We formally state the following,

Proposition 2. A policy π^0 is stochastically optimal for the continuous time problem if and only if the actions prescribed by π^0 constitute a stochastically optimal policy in the discrete time problem.

Proof: It is known, Keilson(1979), that the uniformized process is

6

probabilistically identical to the original process: thus we can assume that $T_\pi^C(s)$ refers to the later. Let $Y_1, Y_2, \ldots$ denote the sequence of i.i.d. sojourn times in the continuous time uniformized process. Then,

$$(7) \quad T_\pi^C(s) = \sum_{k=1}^{T_\pi'(s)} Y_k$$

and the proof can be completed easily: see Ross (1983 p. 255) .

Proposition 2 enables us to study continuous time problems by applying proposition 1 to the discrete time problem obtained via uniformization.

3. Optimal Repair of a Series System.

At any point in time the state of the system is specified by a vector $x = (x_1, \ldots, x_N)$ with the convention that $x_i = 1$ or 0 according to whether the ith component is functioning or not and a scalar θ which denotes the state of the environment. Thus, $S = \{0,1\}^N \times \Theta$ is the set of all possible states and $W = \{(1,\theta) , \theta \in \Theta\}$ is the set of all functioning states for the system.

Given a state $(x,\theta) \in S$, we define:

$C_0(x,\theta) = \{i : x_i = 0\}$,

$C_1(x,\theta) = \{i : x_i = 1\}$,

$(\delta_i, x, \theta) = ((x_1, \ldots, x_{i-1}, \delta, x_{i+1}, \ldots, x_N); \theta)$, for $\delta = 0$ or 1 ,

$a(x; \theta) = \min\{\mu_i(\theta) : i \in C_0(x; \theta)\}$,

$\mu(x; \theta) = \sum_1^N x_i \mu_i(\theta)$,

$q(\theta) = \sum_{\theta'} q(\theta'/\theta)$.

Assumption A: we assume that if $j \leqslant i$ then $\mu_j(\theta) \leqslant \mu_i(\theta)$, $\forall \theta \in \Theta$. This in particular implies that $a(x,\theta) = a(x)$ $\forall (x,\theta) \in S$.

When the system is in state (x,θ) the set of all possible actions can be identified with the set of the failed components with the interpretation that action $i \in C_0(x;\theta)$ means that the repairman is assigned to component i. When the system is in state $(x;\theta;m)$ and action $i \in C_0(x;\theta)$ is chosen the following transitions are possible

i)   to state $(1_i,x;\theta)$, with rate λ_i ,

ii) to state $(0_k,x;\theta)$, with rate $\mu_k(\theta)$, $k \in C_1(x;\theta)$

iii) to state $(x;\theta')$, with rate $q(\theta'/\theta)$, $\theta, \theta' \in \Theta$.

The discrete time decision problem induced by the above is defined on the same state space with the following transition law. When the system is in state $(x;\theta;m)$ and action $i \in C_0(x;\theta)$ is chosen the following transitions are possible.

i)   to state $(1_i,x;\theta)$, with probability λ_i / v ,

ii) to state $(0_k,x;\theta)$, with probability $\mu_k(\theta) / v$, $k \in C_1(x;\theta)$,

iii) to state $(x;\theta')$, with probability $q(\theta'/\theta)/v$, $\theta, \theta' \in \Theta$,

iv) to state $(x;\theta)$, with probability $(v - \mu(x;\theta) - \lambda_i - q(\theta))/v$,

3.1 Construction of a family of Markovian Decision Problems.

We construct a family of discrete time, finite horizon markovian decision models as follows.

For any n positive integer construct problem (Π_n) by defining:

1. States: $(x;\theta;m)$, $(x;\theta) \in \Theta$, $m = 0,1,\ldots,n$

2. Action sets: $A(x;\theta;m) = C_0(x;\theta)$

3. System dynamics: when the system is in state $(x;\theta;m)$ and action $i \in C_0(x;\theta)$ is chosen the following transitions are possible

i)   to state $(1_i,x;\theta;m-1)$, with probability λ_i / v ,

ii) to state $(0_k,x;\theta;m-1)$, with probability $\mu_k(\theta) / v$, $k \in C_1 x;\theta$,

iii) to state $(x;\theta';m-1)$, with probability $q \theta' \theta / v$, $\theta, \theta' \in \Theta$,

8

iv) to state $(x;\theta;m-1)$, with probability $(v - \mu(x;\theta) - \lambda_i - q(\theta))$ v ,

where v is any constant greater than the sum of all transition rates.

4. Reward structure:

$$r(x;\theta;m) = \begin{cases} 1 & \text{if } (x;\theta) \in W \text{ and } m = 0 \\ 0 & \text{otherwise .} \end{cases}$$

3.2. The SFR policy is stochastically optimal.

Take n and denote by $V(x;\theta;m)$ the value of state $(x;\theta;m)$ in the finite horizon problem (Π_n) . To write the functional equations for $V(x;\theta;m)$, $m = 1,\ldots,n$, we first define

$$\Lambda(x;\theta;m;i) = \frac{1}{v} \Big\{ (v - \mu(x;\theta) - \lambda_i - q(\theta)) \ V(x;\theta;m-1)$$

$$+ \lambda_i V(1_i,x;\theta;m-1) + \Sigma_1^N x_k \mu_k(\theta) V(0_k,x;\theta;m-1)$$

$$+ \Sigma_{\theta' \in \Theta} \ q(\theta'/\theta) V(x;\theta';m-1) \Big\}$$

(8) $V(x;\theta;m) = \max_{i \in C_0(x;\theta)} \{\Lambda(x;\theta;m;i)\}$

(9) $V(x;\theta;0) = \begin{cases} 1 & \text{if } (x;\theta) \in W \\ 0 & \text{otherwise .} \end{cases}$

Lemma 3. Let $\Sigma x_i \leq N - 2$. Then under assumption A, for any $(x;\theta) \in S$ i, j $\in C_0(x;\theta)$, $m \geq 1$ and $j \neq i$ the following inequalities hold

(10) $(\lambda_i - \lambda_j) V(x;\theta;m) \geq \lambda_i V(1_i,x;\theta;m) - \lambda_j V(1_j,x;\theta;m)$

(11) $V(x;\theta;m) \leq V(1_i,x;\theta;m)$, $1 \leq i \leq N$.

Proof. We prove the above inequalities simultaneously by induction on m .

Throughout the proof we use the quantities: $L_{ij}(x;\theta;m)$ defined by:

(12) $L_{ij}(x;\theta;m) = (\lambda_i - \lambda_j) V(x;\theta;m) - \lambda_i V(1_i,x;\theta;m) - \lambda_j V(1_j,x;\theta;m)$,

The induction hypothesis is expressed by (13), (14) below

9

(13) $L_{ij}(x;\theta;m) \geq 0$, for all x, θ, i, j   with $j \leq i$

and

(14) $V(x;\theta;m) \leq V(1_i,x;\theta;m)$, $1 \leq i \leq N$.

(13) and (14) are obviously true for m = 1. We next complete the induction as follows.

Step 1: we first show that the following inequalities hold.

(15) $(\lambda_i - \lambda_j)((v - \lambda_{a(x)} - \mu(x;\theta) - q(\theta))V(x;\theta;m) + \lambda_{a(x)}V(1_{a(x)},x;\theta;m)) \geq$

$\geq \lambda_i((v - \lambda_{a(x)} - \mu_i(\theta) - \mu(x;\theta) - q(\theta))V(1_i,x;\theta;m)$

$- \lambda_{a(x)}V(1_i,1_{a(x)},x;\theta;m) + \mu_i(\theta)V(x;\theta;m))$

$- \lambda_j(v - \lambda_{\ell(x)} - \mu_j(\theta) - \mu(x;\theta) - q(\theta))V(1_j,x;\theta;m)$

$- \lambda_{\ell(x)}V(1_{\ell(x)},1_j,x;\theta;m) - \mu_j(\theta)V(x;\theta;m))$

where $\ell(x) = \begin{cases} i & \text{if } j = a(x) \\ a(x) & \text{if } j > a(x) \end{cases}$

To establish (15) we first note that

(16) $-\lambda_j L_{ij}(x;\theta;m) = (\lambda_i - \lambda_j)(-\lambda_j V(x;\theta;m) + \lambda_j V(1_j,x;\theta;m))$

$-\lambda_i(-\lambda_j V(1_i,x;\theta;m) + \lambda_j V(1_i,1_j,x;\theta;m))$

$+\lambda_j(-\lambda_i V(1_j,x;\theta;m) + \lambda_i V(1_i,1_j,x;\theta;m))$

We also note that from (13) we have

(17) $\lambda_{a(x)}L_{ij}(1_{a(x)}x;\theta) \geq 0$

Next (13) implies that

(18) $(v - \lambda_{a(x)} - \mu_j(\theta) - \mu(x;\theta) - q(\theta))L_{ij}(x;\theta;m) \geq 0$

which leads to

(19) $(v - \mu(x;\theta) - q(\theta))L_{ij}(x;\theta;m) - \lambda_a L_{ij}(x;\theta;m) \geq$

$$\geq \mu_j(\theta;\lambda_i(V(x;\theta;m) - V(1_i,x;\theta;m)) - \mu_j(\theta)\lambda_j(V(x;\theta;m) - V(1_j,x;\theta;m))$$

$$\geq \mu_i(\theta)\lambda_i(V(x;\theta;m) - V(1_i,x;\theta;m)) - \mu_j(\theta)\lambda_j(V(x;\theta;m) - V(1_j,x;\theta;m))$$

Where the last inequality follows from the induction hypothesis (11) and the fact that $\mu_j(\theta) \leq \mu_i(\theta)$.

Using (16) or (17) in (19) (according to whether $a(x) = j$ or $a(x) \neq j$) we obtain (15), after simple operations.

Step 2: we next show that the following inequality holds.

$$(20) \quad (\lambda_i - \lambda_j) \textstyle\sum_{k=1}^{N} x_k \mu_k(\theta) V(0_k,x;\theta;m)$$

$$\geq \lambda_i \textstyle\sum_{k=1}^{N} x_k \mu_k(\theta) V(1_i,0_k,x;\theta;m) - \lambda_j \textstyle\sum_{k=1}^{N} x_k \mu_k(\theta) V(1_j,0_k,x;\theta;m)$$

To prove (20) it suffices to note that (10) implies that:

$$(21) \quad \textstyle\sum_{k=1}^{N} x_k \mu_k L_{ij}(0_k,x;\theta;m) \geq 0 , \text{ for all } x, \theta, i, j .$$

Step 3: the next inequality is established.

$$(22) \quad (\lambda_i - \lambda_j) \textstyle\sum_{\theta'} q(\theta'/\theta) V(x;\theta';m)$$

$$\geq \lambda_i \textstyle\sum_{\theta'} q(\theta'/\theta) V(1_i,x;\theta';m) - \lambda_j \textstyle\sum_{\theta'} q(\theta'/\theta) V(1_j,x;\theta';m)$$

To prove (22) it suffices to note that (10) implies that:

$$(23) \quad \textstyle\sum_{\theta'} q(\theta'/\theta) L_{ij}(x;\theta';m) \geq 0 , \text{ for all } x, \theta', i, j .$$

Notice that for (23) to hold it essential that the failure rates of the components of the system have the same ordering for all θ (assumption A) and that the repair rates are independent of θ .

Step 4: we next complete the induction step for inequalities (10) .

We multiply both sides of inequalities (15), (20), (22) by $\frac{1}{v}$ and add them. Thus, after some simple algebra we obtain the following inequality.

$$(24) \quad (\lambda_i - \lambda_j)\Lambda(x;\theta;m;a(x)) \geq \lambda_i\Lambda(1_i,x;\theta;m;a(x)) - \lambda_j\Lambda(1_j,x;\theta;m;a(x))$$

Now observe that the following relations hold due to the inductive hypothesis.

$$(25) \quad \Lambda'x;\theta:m:a(x)) = V(x:\theta:m+1)$$

$$(26) \quad \Lambda(1_i,x;\theta:m:a(x)) = V(1_i,x;\theta:m+1)$$

$$(27) \quad \Lambda(1_j,x:\theta:m:\ell(x)) \leq V(1_j,x:\theta:m+1)$$

Therefore, (10) holds for $m+1$ also.

Step 5: we complete the induction step for relation (11).

Let $$b(x) = \begin{cases} a(1_i,x) & \text{if } i = a(x) \\ a(x) & \text{if } i > a(x) \end{cases}$$

After simple computations in (11) for $m+1$ the reader may check that it suffices to establish

$$(28) \quad (v - \lambda_{a(x)} - \mu_i(\theta) - \mu(x;\theta) - q(\theta))(V(1_i,x:\theta:m) - V(x;\theta:m)) \geq$$

$$\lambda_{b(x)}(V(1_{a(x)},x;\theta:m) - V(1_{b(x)}, 1_i,x:\theta:m)) +$$

$$\sum_{k=1}^{N} x_k \mu_k(\theta)(V(0_k,x;\theta:m) - V(1_i,0_k,x;\theta:m))$$

$$\sum_{\theta'} q(\theta'/\theta)(V(x;\theta';m) - V(1_i,x;\theta';m))$$

Now, (28) holds since, from the inductive hypothesis, the left hand side of (28) is nonnegative while the right hand side is nonpositive: note that the first term in the right hand side of (28) is always a difference of the type covered by the induction hypothesis (14).

Theorem 1. Under the assumptions made, the SFR policy minimizes stochastically the time the system spends under repair .

Proof. The previous lemma shows that the SFR policy is optimal for all (Π_n) $n \geq 1$. Hence the result follows from Propositions 1 and 2 .

12

4. Scheduling Jobs to Minimize The Makespan.

The particular problem we examine is that considered in Van der Heyden (1981). The state space for this problem is: $S = \{x : x = \{x_1, x_2, \ldots, x_m\} , m = 1, \ldots , x_i \in (0, B) \} \cup \emptyset$, where state $\emptyset$ is the empty state for the system , x_i denotes the processing rate of the i-th job in the system (waiting or being processed), and B is some bound. The action set in state x , $A(x)$, contains all subsets of x that contain at most p elements.

After uniformization, the functional equations for the finite horizon problem we consider can be writen as :

$$(29) \quad V(x;m+1) = \max_{a \in A(x)} \{ [\Sigma_{j \in a} V(x \backslash \{x_j\}; m) x_j + r \, E(v(x \cup \{y\}; m)] / v \} , \quad 1 \leq m \leq n$$

with boundary condition,

$$(30) \quad V(x;0) = \begin{cases} 1 & \text{if} \quad x = \emptyset \\ 0 & \text{otherwise .} \end{cases}$$

where the expectation is taken with respect to the random processing rate y chosen from the distribution F.

We can readily establish the following,

Theorem 2. The LEPT policy is stochastically optimal.

Proof: To estabilsh the stochastic optimality of the LEPT policy, it suffices to show that it is optimal in the finite horizon problem defined above for all n . The proof of this is equivalent to establishing inequalities which are analogous to those in part 3 of Van der Heyden (1981), i.e. inequalities $(3.2)_n$, $(3.3)_n$, $(3.4)_n$ but with the inequality sign reversed. This can be done by examining all cases as in Van der Heyden (1981) and following exactly the same steps but with symmetric arguments.

13

REFERENCES.

Derman, C. (1970). *"Finite State Markovian Decision Processes"*, Academic

Press, New York.

Jensen A. (1953). "Markov Chains as an aid in the study of Markov Processes",

Skand. Aktuarietidskr. No 36, 87-91 .

Katehakis M. N. and C. Derman (1984). "Optimal repair allocation in a Series

System", *Mathematics of Operations Research*, 9, No. 4, 615-623 .

Keilson(1979). *Markov Chain Models - Rarity and exponentiality,*
Springer-Verlag, New York.

Lippman S. A. (1975). "Applying a new device in the optimization of exponential

Queuing Systems", *Operations Research*, 23, 687-710.

Smith D. R. (1978). "Optimal Repair of a Series System", *Operations Res.,*

26, 653-662 .

Ross, S. M. (1983). *"Stochastic Processes"*, Willey , New York.

Van der Heyden L. (1981). "Scheduling jobs with exponential processing and

arrival times on identical processors so as to minimize the expected

makespan, *Mathematics of Operations Research*, 6, No. 2, 305-312 .

Veinott A. F. Jr. (1969), "Discrete dynamic programming with sensitive discount

optimality criteria", *Annals of Mathematical Statistics*, 40, 1635-1660.

Weiss G. and Pinedo M. (1980) . "Scheduling tasks with exponential service

times on nonidentical processors to minimize various cost functions",
Journal of Applied Probability, No 17, 187-202 .

END
DATE
FILMED
JAN
1988